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Integrable semi-discretization of the coupled nonlinear Schrödinger equations

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Abstract. A system of semi-discrete coupled nonlinear Schrödinger equations is studied. To show the complete integrability of the model with multiple components, we extend the discrete version of the inverse scattering method for the single-component discrete nonlinear Schrödinger equation proposed by Ablowitz and Ladik. By means of the extension, the initial-value problem of the model is solved. Further, the integrals of motion and the soliton solutions are constructed within the framework of the extension of the inverse scattering method.

1. Introduction

There has been a surge of interest in the family of nonlinear Schrödinger (NLS) equations because of its many applications to various kinds of physical phenomena. In the remarkable papers [1, 2] Zakharov and Shabat solved the NLS model,

$$\begin{aligned} &\mathrm{i}\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} - 2qrq = 0 \\ &\mathrm{i}\frac{\partial r}{\partial t} - \frac{\partial^2 r}{\partial x^2} + 2rqr = 0 \end{aligned} \tag{1.1}$$

with $r = \mp q^*$ by means of the inverse scattering method (ISM). After another success of the ISM for the modified KdV equation [3], Ablowitz, Kaup, Newell and Segur [4] unified a class of soliton equations by employing various time dependences of the scattering problem. Their formulation is called the AKNS formulation.

A number of authors have studied extensions of the AKNS formulation and presented many models, which are integrable by the ISM [5, 6]. Among such models, a system of coupled nonlinear Schrödinger (CNLS) equations

$$\begin{aligned} &\mathrm{i}\frac{\partial q_j}{\partial t} + \frac{\partial^2 q_j}{\partial x^2} - 2\sum_{k=1}^{m} q_k r_k\, q_j = 0 \\ &\mathrm{i}\frac{\partial r_j}{\partial t} - \frac{\partial^2 r_j}{\partial x^2} + 2\sum_{k=1}^{m} r_k q_k\, r_j = 0 \end{aligned} \qquad j = 1, 2, \ldots, m \tag{1.2}$$

is particularly remarkable in describing diverse physical phenomena [7–19]. Manakov [8] considered the two-component CNLS equations (equation (1.2) with $m = 2$) with $r_j = -q_j^*$ $(j = 1, 2)$ as a model for propagation of two polarized electromagnetic waves

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and applied the ISM to the model for the first time. Interest has recently focused on the two-component CNLS equations in studying explicit solutions [10, 11], the stability of solitary waves [14] and interactions between solitons in birefringent optical fibres [18, 19] from a physical point of view.

Very recently, the authors proposed a new extension of the ISM and solved the coupled modified KdV (cmKdV) equations [20]

$$\frac{\partial u_i}{\partial t} + 6\left(\sum_{j,k=0}^{M-1} C_{jk} u_j u_k\right)\frac{\partial u_i}{\partial x} + \frac{\partial^3 u_i}{\partial x^3} = 0 \qquad i = 0, 1, \ldots, M-1 \tag{1.3}$$

in the self-focusing case, which had been investigated by alternative approaches [21, 22]. By a transformation of variables, this model is cast into a new coupled version of the Hirota equation [23], which describes wave propagation in optical fibres, including higher-order effects.

On the other hand, some discrete versions of the ISM have been constructed and applied to some discrete models [24–29]. Among those models, the semi-discrete nonlinear Schrödinger (sd-NLS) equation found by Ablowitz and Ladik [28]

$$\begin{aligned} &\mathrm{i}\frac{\partial q_n}{\partial t} + (q_{n+1} + q_{n-1} - 2q_n) - q_n r_n (q_{n+1} + q_{n-1}) = 0 \\ &\mathrm{i}\frac{\partial r_n}{\partial t} - (r_{n+1} + r_{n-1} - 2r_n) + r_n q_n (r_{n+1} + r_{n-1}) = 0 \end{aligned} \tag{1.4}$$

has been studied extensively because of its simplicity and physical significance. They solved (1.4) under the rapidly decreasing boundary conditions, $q_n, r_n \to 0$ at $n \to \pm\infty$. The model (1.4) was also solved under other integrable boundary conditions [30–32].

The sd-NLS equation has attracted researchers to studies on various subjects, such as nonlinear lattices in condensed matter physics [33], phase plane patterns [34], breather solutions [35], Bäcklund transformations [36], numerical experiments and homoclinic structure [37–39], the dynamics of a discrete curve [40] and surfaces [41], Hamiltonian structure and the classical r-matrix representation [42–44] and the quantization of the model [42].

In an analogous way to the continuous theory, it is natural to consider a generalization of the sd-NLS equation (1.4) with multiple components, namely,

$$\left.\begin{aligned} &\mathrm{i}\frac{\partial q_n^{(j)}}{\partial t} + \left(q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}\right) - \sum_{k=1}^{m} q_n^{(k)} r_n^{(k)} \left(q_{n+1}^{(j)} + q_{n-1}^{(j)}\right) = 0 \\ &\mathrm{i}\frac{\partial r_n^{(j)}}{\partial t} - \left(r_{n+1}^{(j)} + r_{n-1}^{(j)} - 2r_n^{(j)}\right) + \sum_{k=1}^{m} r_n^{(k)} q_n^{(k)} \left(r_{n+1}^{(j)} + r_{n-1}^{(j)}\right) = 0 \end{aligned}\right. \qquad j = 1, 2, \ldots, m. \tag{1.5}$$

We call this model the semi-discrete coupled NLS (sd-CNLS) equations. The model is expected to be important in various applications, e.g. numerical simulations of the CNLS equations (1.2). Hisakado [45] showed that the system (1.5) is connected with the two-dimensional Toda lattice. An N-soliton solution was obtained by Ohta [46]. It is noted that another scheme of integrable semi-discretization of the CNLS equations was reported by Merola *et al* [47].

In [48], the authors proposed a new extension of the discrete version of the ISM by Ablowitz and Ladik. Applying the extension, they solved the initial-value problem of the semi-discrete coupled modified KdV (sd-cmKdV) equations, or the coupled modified Volterra equations [45],

$$\frac{\partial u_n^{(i)}}{\partial t} = \left(1 + \sum_{j,k=0}^{M-1} C_{jk} u_n^{(j)} u_n^{(k)}\right)\left(u_{n+1}^{(i)} - u_{n-1}^{(i)}\right) \qquad i = 0, 1, \ldots, M-1 \tag{1.6}$$

under some appropriate conditions. A systematic procedure for constructing conservation laws and multi-soliton solutions was also given [48]. Related results are obtained by means of Hirota's method [46, 49].

In the present paper, we use a transformation of variables

$$\begin{aligned} \mathrm{i}^n \mathrm{e}^{2\mathrm{i}t} q_n^{(j)} &= v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} \\ (-\mathrm{i})^n \mathrm{e}^{-2\mathrm{i}t} r_n^{(j)} &= -v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} \end{aligned} \qquad j = 1, 2, \ldots, m \tag{1.7}$$

which cast the sd-CNLS equations (1.5) into the sd-cmKdV equations (1.6) with $C_{jk} = \delta_{j,k}$, $u_n^{(i)} \to v_n^{(i)}$ and $M = 2m$. We pull back the transformation (1.7) to the level of the Lax representation and give an explicit Lax pair for the sd-CNLS equations (1.5) for the first time. Following the method of [48], we can solve the initial-value problem of the sd-CNLS equations (1.5) with $r_n^{(j)} = -q_n^{(j)*}$ under the rapidly decreasing boundary conditions, $q_n^{(j)} \to 0$ at $n \to \pm\infty$. Explicit forms of conserved quantities and the N-soliton solution are also given within the framework of the ISM.

The paper consists of the following. In section 2, we introduce a Lax pair for the semi-discrete matrix NLS equation. Considering a reduction to the sd-CNLS equations, we obtain the Lax formulation and conservation laws for the sd-CNLS equations. In section 3, we perform the ISM for the sd-CNLS equations with directing our attention to the transformation (1.7). The initial-value problem is solved and the N-soliton solution is given. The last section is devoted to discussions.

The main idea of the paper is based on a matrix representation and some properties of the Clifford algebra, whose elements are anti-commutative. The proof of relations used in the paper is given in [48].

2. Lax representation and conservation laws

2.1. Lax pair for the semi-discrete matrix NLS equation

We begin with a set of auxiliary linear equations

$$\Psi_{n+1} = L_n \Psi_n \qquad \Psi_{n,t} = M_n \Psi_n. \tag{2.1}$$

Here Ψ_n is a $2l$-component column vector, and L_n, M_n are $2l \times 2l$ matrices. The compatibility condition of (2.1) is given by

$$L_{n,t} + L_n M_n - M_{n+1} L_n = O. \tag{2.2}$$

We call L_n and M_n the Lax pair and (2.2) (a semi-discrete version of) the zero-curvature condition, or simply, the Lax equation. Let us introduce the following form for the Lax pair:

$$L_n = z \begin{bmatrix} F_1 & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & F_1 Q_n \\ F_2 R_n & O \end{bmatrix} + \frac{1}{z} \begin{bmatrix} O & O \\ O & F_2 \end{bmatrix} = \begin{bmatrix} z F_1 & F_1 Q_n \\ F_2 R_n & F_2 / z \end{bmatrix} \tag{2.3}$$

$$\begin{aligned} M_n = {} & z^2 \begin{bmatrix} \mathrm{i} I & O \\ O & O \end{bmatrix} + z \begin{bmatrix} O & \mathrm{i} Q_n \\ \mathrm{i} F_2 R_{n-1} F_1 & O \end{bmatrix} \\ & + \begin{bmatrix} -\mathrm{i} Q_n F_2 R_{n-1} F_1 + \mathrm{i} H_1 & O \\ O & \mathrm{i} R_n F_1 Q_{n-1} F_2 + \mathrm{i} H_2 \end{bmatrix} \\ & + \frac{1}{z} \begin{bmatrix} O & -\mathrm{i} F_1 Q_{n-1} F_2 \\ -\mathrm{i} R_n & O \end{bmatrix} + \frac{1}{z^2} \begin{bmatrix} O & O \\ O & -\mathrm{i} I \end{bmatrix} \\ = {} & \mathrm{i} \begin{bmatrix} z^2 I - Q_n F_2 R_{n-1} F_1 + H_1 & z Q_n - (1/z) F_1 Q_{n-1} F_2 \\ z F_2 R_{n-1} F_1 - R_n / z & -I/z^2 + R_n F_1 Q_{n-1} F_2 + H_2 \end{bmatrix} \end{aligned} \tag{2.4}$$

where z is the spectral parameter which is time independent. I is the $l \times l$ unit matrix, Q_n and R_n are $l \times l$ matrices. The constant matrices F_1, F_2, H_1 and H_2 are assumed to be Hermitian and satisfy

$$(F_1)^2 = (F_2)^2 = I \qquad [F_1, H_1] = [F_2, H_2] = O. \tag{2.5}$$

Here $[\cdot\,,\cdot]$ denotes the commutator. Substituting (2.3) and (2.4) into (2.2), we obtain a set of matrix equations

$$\begin{aligned} &\mathrm{i}Q_{n,t} + F_1(Q_{n+1} + Q_{n-1})F_2 + H_1 Q_n - Q_n H_2 - F_1 Q_{n+1} F_2 R_n Q_n \\ &\qquad -Q_n R_n F_1 Q_{n-1} F_2 = O \\ &\mathrm{i}R_{n,t} - F_2(R_{n+1} + R_{n-1})F_1 - R_n H_1 + H_2 R_n + F_2 R_{n+1} F_1 Q_n R_n \\ &\qquad +R_n Q_n F_2 R_{n-1} F_1 = O. \end{aligned} \tag{2.6}$$

We call this model the semi-discrete (sd-) matrix NLS equation. The integrable model (2.6) with $F_1 = F_2 = I$, $H_1 = -I$, $H_2 = I$ was found by Ablowitz *et al* [50, 51].

2.2. Conservation laws

In this subsection, we present a method to construct local conservation laws for the sd-matrix NLS equation (2.6), which is a discrete version of the method in the continuous theory [20]. We start from an explicit expression of (2.1),

$$\begin{bmatrix} \Psi_{1\,n+1} \\ \Psi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} F_{1\,n} & S_n \\ T_n & F_{2\,n} \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix} \qquad \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix}_t = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix} \tag{2.7}$$

where all the entries in vectors and matrices are assumed to be $l \times l$ square matrices.

Introducing an $l \times l$ square matrix Γ_n by

$$\Gamma_n \equiv \Psi_{2\,n}\Psi_{1\,n}^{-1} \tag{2.8}$$

we can show the following relations from (2.2) and (2.7) [48]:

$$\begin{aligned} &(S_n\Gamma_n + F_{1\,n})_t(S_n\Gamma_n + F_{1\,n})^{-1} = A_{n+1} - (S_n\Gamma_n + F_{1\,n})A_n(S_n\Gamma_n + F_{1\,n})^{-1} \\ &\qquad +B_{n+1}\Gamma_{n+1} - (S_n\Gamma_n + F_{1\,n})B_n\Gamma_n(S_n\Gamma_n + F_{1\,n})^{-1} \end{aligned} \tag{2.9}$$

$$\Gamma_{n+1} = (T_n + F_{2\,n}\Gamma_n)(F_{1\,n} + S_n\Gamma_n)^{-1}. \tag{2.10}$$

Taking the trace on both sides of (2.9), we obtain

$$\mathrm{tr}\{\log(S_n\Gamma_n + F_{1\,n})\}_t = \mathrm{tr}(A_{n+1} + B_{n+1}\Gamma_{n+1}) - \mathrm{tr}(A_n + B_n\Gamma_n). \tag{2.11}$$

Assuming the form of L_n as (2.3), we have

$$F_{1\,n} = zF_1 \qquad F_{2\,n} = \frac{1}{z}F_2 \qquad S_n = F_1 Q_n \qquad T_n = F_2 R_n. \tag{2.12}$$

Then (2.11) and (2.10) are transformed into

$$\left\{\mathrm{tr}\ \log\left(I + \frac{1}{z}Q_n\Gamma_n\right)\right\}_t = \mathrm{tr}(A_{n+1} + B_{n+1}\Gamma_{n+1}) - \mathrm{tr}(A_n + B_n\Gamma_n) \tag{2.13}$$

$$zQ_n\Gamma_n = Q_n F_2 R_{n-1} F_1 + \frac{1}{z} Q_n F_2 Q_{n-1}^{-1}(Q_{n-1}\Gamma_{n-1})F_1 - (Q_n\Gamma_n)F_1(Q_{n-1}\Gamma_{n-1})F_1. \tag{2.14}$$

Note that (2.13) has the form of the local conservation law. This suggests that $\mathrm{tr}\{\log(I + Q_n\Gamma_n/z)\}$ is a generator of the conserved densities for (2.6). We substitute the expansion of $Q_n\Gamma_n$ with respect to $1/z$,

$$Q_n\Gamma_n^{(-)} = \sum_{j=1}^{\infty} \frac{1}{z^{2j-1}} f_n^{(j)} \tag{2.15}$$

into (2.14). Then we obtain a recursion formula for $f_n^{(j)}$,

$$f_n^{(j)} = Q_n F_2 R_{n-1} F_1 \delta_{j,1} + Q_n F_2 Q_{n-1}^{-1} f_{n-1}^{(j-1)} F_1 - \sum_{k=1}^{j-1} f_n^{(k)} F_1 f_{n-1}^{(j-k)} F_1 \qquad j = 1, 2, \ldots. \tag{2.16}$$

Equation (2.16) yields $f_n^{(j)}$, for instance,

$$f_n^{(1)} = Q_n F_2 R_{n-1} F_1$$
$$f_n^{(2)} = Q_n R_{n-2} - Q_n F_2 R_{n-1} Q_{n-1} F_2 R_{n-2}.$$

We substitute (2.15) into $\mathrm{tr}\{\log(I + Q_n \Gamma_n^{(-)}/z)\}$ and expand it with respect to $1/z$,

$$\mathrm{tr}\left\{\log\left(I + \frac{1}{z^2} f_n^{(1)} + \frac{1}{z^4} f_n^{(2)} + \frac{1}{z^6} f_n^{(3)} + \cdots\right)\right\} = \mathrm{tr}\left\{\frac{1}{z^2} f_n^{(1)} + \frac{1}{z^4}\left[f_n^{(2)} - \tfrac{1}{2}(f_n^{(1)})^2\right] + \cdots\right\}.$$

Thus, the first two conserved densities given by this expansion are

$$J_n^{(-1)} = \mathrm{tr}\left\{f_n^{(1)}\right\} = \mathrm{tr}\{Q_n F_2 R_{n-1} F_1\} \tag{2.17a}$$

$$\begin{aligned} J_n^{(-2)} &= \mathrm{tr}\left\{f_n^{(2)} - \tfrac{1}{2}(f_n^{(1)})^2\right\} \\ &= \mathrm{tr}\left\{Q_n R_{n-2} - Q_n F_2 R_{n-1} Q_{n-1} F_2 R_{n-2} - \tfrac{1}{2}(Q_n F_2 R_{n-1} F_1)^2\right\}. \end{aligned} \tag{2.17b}$$

Similarly, we expand $Q_n \Gamma_n$ with respect to z,

$$Q_n \Gamma_n^{(+)} = \sum_{j=1}^{\infty} z^{2j-1} g_n^{(j)}. \tag{2.18}$$

Substitution of (2.18) into (2.14) yields a recursion formula for $g_n^{(j)}$,

$$g_n^{(j)} = -Q_n R_n \delta_{j,1} + Q_n F_2 Q_{n+1}^{-1} g_{n+1}^{(j-1)} F_1 + Q_n F_2 Q_{n+1}^{-1} \sum_{k=1}^{j-1} g_{n+1}^{(k)} F_1 g_n^{(j-k)} \qquad j = 1, 2, \ldots. \tag{2.19}$$

From formula (2.19), the first three of the coefficients $g_n^{(j)}$ are given by

$$\begin{aligned} g_n^{(1)} &= -Q_n R_n \\ g_n^{(2)} &= -Q_n F_2 R_{n+1} F_1 (I - Q_n R_n) \\ g_n^{(3)} &= -Q_n R_{n+2}(I - Q_n R_n) + (Q_n F_2 R_{n+1} F_1)^2 (I - Q_n R_n) \\ &\quad + Q_n R_{n+2} F_1 Q_{n+1} R_{n+1} F_1 (I - Q_n R_n). \end{aligned}$$

We substitute (2.18) into $\mathrm{tr}\{\log(I + Q_n \Gamma_n^{(+)}/z)\}$ and expand it with respect to z,

$$\begin{aligned} \mathrm{tr}\left\{\log\left(I + g_n^{(1)} + z^2 g_n^{(2)} + z^4 g_n^{(3)} + \cdots\right)\right\} &= \mathrm{tr}\Big\{\log\left(I + g_n^{(1)}\right) + z^2 g_n^{(2)}\left(I + g_n^{(1)}\right)^{-1} \\ &\quad + z^4\left[g_n^{(3)}\left(I + g_n^{(1)}\right)^{-1} - \tfrac{1}{2}\left\{g_n^{(2)}\left(I + g_n^{(1)}\right)^{-1}\right\}^2\right] + \cdots\Big\}. \end{aligned}$$

Thus, the first three conserved densities in this expansion are

$$J_n^{(0)} = \mathrm{tr}\left\{\log\left(I + g_n^{(1)}\right)\right\} = \mathrm{tr}\left\{\log\left(I - Q_n R_n\right)\right\} \tag{2.20a}$$

$$J_n^{(1)} = \mathrm{tr}\left\{g_n^{(2)}\left(I + g_n^{(1)}\right)^{-1}\right\} = \mathrm{tr}\left\{-Q_n F_2 R_{n+1} F_1\right\} \tag{2.20b}$$

$$\begin{aligned} J_n^{(2)} &= \mathrm{tr}\left[g_n^{(3)}\left(I + g_n^{(1)}\right)^{-1} - \tfrac{1}{2}\left\{g_n^{(2)}\left(I + g_n^{(1)}\right)^{-1}\right\}^2\right] \\ &= \mathrm{tr}\left\{-Q_n R_{n+2} + Q_n R_{n+2} F_1 Q_{n+1} R_{n+1} F_1 + \tfrac{1}{2}(Q_n F_2 R_{n+1} F_1)^2\right\}. \end{aligned} \tag{2.20c}$$

The generator of the conserved densities, $\mathrm{tr}\{\log(I + Q_n \Gamma_n/z)\}$, is shown to be related with a time-independent subset of scattering data defined later (see the appendix).

2.3. Reduction of the Lax pair and the conservation laws for the semi-discrete coupled NLS equations

In this subsection, we show a reduction of the sd-matrix NLS equations to the sd-CNLS equations. We recursively define $2^{m-1} \times 2^{m-1}$ matrices $F_1^{(m)}$, $F_2^{(m)}$, $H_1^{(m)}$, $H_2^{(m)}$, $Q_n^{(m)}$ and $R_n^{(m)}$ by

$$F_1^{(1)} = 1 \quad F_2^{(1)} = 1 \quad H_1^{(1)} = -1 \quad H_2^{(1)} = 1 \tag{2.21}$$

$$F_1^{(m+1)} = \begin{bmatrix} F_1^{(m)} & \\ & -F_2^{(m)} \end{bmatrix} \qquad F_2^{(m+1)} = \begin{bmatrix} F_2^{(m)} & \\ & F_1^{(m)} \end{bmatrix} \tag{2.22}$$

$$H_1^{(m+1)} = \begin{bmatrix} H_1^{(m)} - I_{2^{m-1}} & \\ & H_2^{(m)} + I_{2^{m-1}} \end{bmatrix} \tag{2.23}$$

$$H_2^{(m+1)} = \begin{bmatrix} H_2^{(m)} - I_{2^{m-1}} & \\ & H_1^{(m)} + I_{2^{m-1}} \end{bmatrix} \tag{2.24}$$

$$Q_n^{(1)} = q_n^{(1)} \qquad R_n^{(1)} = r_n^{(1)} \tag{2.25}$$

$$Q_n^{(m+1)} = \begin{bmatrix} Q_n^{(m)} & q_n^{(m+1)} I_{2^{m-1}} \\ r_n^{(m+1)} I_{2^{m-1}} & -R_n^{(m)} \end{bmatrix} \qquad R_n^{(m+1)} = \begin{bmatrix} R_n^{(m)} & q_n^{(m+1)} I_{2^{m-1}} \\ r_n^{(m+1)} I_{2^{m-1}} & -Q_n^{(m)} \end{bmatrix}. \tag{2.26}$$

Here $I_{2^{m-1}}$ is the $2^{m-1} \times 2^{m-1}$ unit matrix. It is readily seen that (2.5) is satisfied. For the matrices defined by (2.21)–(2.26), we can prove the following relations:

$$Q_n^{(m)} R_n^{(m)} = R_n^{(m)} Q_n^{(m)} = \sum_{j=1}^{m} q_n^{(j)} r_n^{(j)} \, I_{2^{m-1}} \tag{2.27}$$

$$H_1^{(m)} Q_n^{(m)} - Q_n^{(m)} H_2^{(m)} = -2 F_1^{(m)} Q_n^{(m)} F_2^{(m)}$$

$$-R_n^{(m)} H_1^{(m)} + H_2^{(m)} R_n^{(m)} = 2 F_2^{(m)} R_n^{(m)} F_1^{(m)}$$

by induction. Then substituting $Q_n^{(m)}$, $R_n^{(m)}$, etc. into Q_n, R_n, etc. in the sd-matrix NLS equation (2.6), we obtain the sd-CNLS equations

$$\begin{aligned} &\mathrm{i}\frac{\partial q_n^{(j)}}{\partial t} + (q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}) - \sum_{k=1}^{m} q_n^{(k)} r_n^{(k)} \, (q_{n+1}^{(j)} + q_{n-1}^{(j)}) = 0 \\ &\mathrm{i}\frac{\partial r_n^{(j)}}{\partial t} - (r_{n+1}^{(j)} + r_{n-1}^{(j)} - 2r_n^{(j)}) + \sum_{k=1}^{m} r_n^{(k)} q_n^{(k)} \, (r_{n+1}^{(j)} + r_{n-1}^{(j)}) = 0 \end{aligned} \qquad j = 1, 2, \ldots, m. \tag{2.28}$$

For instance, the Lax matrix L_n for the two-component sd-CNLS equations ((2.28) with $m = 2$) is given by

$$L_n = \begin{bmatrix} z & 0 & q_n^{(1)} & q_n^{(2)} \\ 0 & -z & -r_n^{(2)} & r_n^{(1)} \\ r_n^{(1)} & q_n^{(2)} & 1/z & 0 \\ r_n^{(2)} & -q_n^{(1)} & 0 & 1/z \end{bmatrix}. \tag{2.29}$$

In what follows, we set $r_n^{(j)} = -q_n^{(j)*}$ $(j = 1, 2, \ldots, m)$ and consider a self-focusing case of the sd-CNLS equations,

$$\mathrm{i}\frac{\partial q_n^{(j)}}{\partial t} + (q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}) + \sum_{k=1}^{m} |q_n^{(k)}|^2 \, (q_{n+1}^{(j)} + q_{n-1}^{(j)}) = 0 \qquad j = 1, 2, \ldots, m. \tag{2.30}$$

In this case, the relation (2.27) for $Q_n^{(m)}$ and $R_n^{(m)}$ becomes

$$Q_n^{(m)} R_n^{(m)} = R_n^{(m)} Q_n^{(m)} = -\sum_{j=1}^{m} |q_n^{(j)}|^2 \, I_{2^{m-1}}. \tag{2.31}$$

In addition, a simple relation between $Q_n^{(m)}$ and $R_n^{(m)}$ holds,

$$R_n^{(m)} = -Q_n^{(m)\dagger} \tag{2.32}$$

where the symbol † denotes the Hermitian conjugate. The relations (2.31) and (2.32) play an essential role in the ISM.

Because the sd-CNLS equations are given as a reduction of the sd-matrix NLS equation (2.6), the results in section 2.2 assure the existence of an infinite number of conservation laws for the sd-CNLS equations. Explicit forms of the first four conserved densities for the sd-CNLS equations (2.28) are given by

$$\log\left(1 - \sum_j q_n^{(j)} r_n^{(j)}\right) \tag{2.33}$$

$$\begin{aligned}
&\mathrm{e}^{4\mathrm{i}t}(-1)^n \left(q_{n+1}^{(j)} q_n^{(k)} - q_n^{(j)} q_{n+1}^{(k)}\right) && \forall\ j,\ k\\
&\mathrm{e}^{-4\mathrm{i}t}(-1)^n \left(r_{n+1}^{(j)} r_n^{(k)} - r_n^{(j)} r_{n+1}^{(k)}\right) && \forall\ j,\ k\\
&q_{n+1}^{(j)} r_n^{(k)} + q_n^{(j)} r_{n+1}^{(k)} && \forall\ j,\ k\\
&\sum_j q_{n+1}^{(j)} r_n^{(j)} \qquad \sum_j q_n^{(j)} r_{n+1}^{(j)}
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
&\left(1 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)}\right) \sum_j \left(q_{n+2}^{(j)} r_n^{(j)} + q_n^{(j)} r_{n+2}^{(j)}\right)\\
&\qquad -\tfrac{1}{2}\left\{\sum_j \left(q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}\right)\right\}^2 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)} \sum_j q_n^{(j)} r_n^{(j)}
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
&\left(1 - \sum_j q_{n+2}^{(j)} r_{n+2}^{(j)}\right)\left(1 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)}\right) \sum_j \left(q_{n+3}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+3}^{(j)}\right) - \left(1 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)}\right)\\
&\qquad \times \sum_j \left(q_{n+2}^{(j)} r_n^{(j)} + q_n^{(j)} r_{n+2}^{(j)}\right)\left\{\sum_j \left(q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}\right)\right.\\
&\qquad \left. + \sum_j \left(q_{n+2}^{(j)} r_{n+1}^{(j)} - q_{n+1}^{(j)} r_{n+2}^{(j)}\right)\right\} - \left(1 - \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)}\right)\left\{\sum_j \left(q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}\right)\right.\\
&\qquad \left. \times \sum_j q_{n+2}^{(j)} r_{n+2}^{(j)} + \sum_j \left(q_{n+2}^{(j)} r_{n+1}^{(j)} - q_{n+1}^{(j)} r_{n+2}^{(j)}\right) \sum_j q_n^{(j)} r_n^{(j)}\right\}\\
&\qquad + \tfrac{1}{3}\left\{\sum_j \left(q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}\right)\right\}^3\\
&\qquad + \sum_j \left(q_{n+1}^{(j)} r_n^{(j)} - q_n^{(j)} r_{n+1}^{(j)}\right) \sum_j q_{n+1}^{(j)} r_{n+1}^{(j)} \sum_j q_n^{(j)} r_n^{(j)}.
\end{aligned} \tag{2.36}$$

A straightforward calculation shows that all the entries in (2.34) are conserved densities. It is noted that the lower two densities in (2.34) correspond to the conserved densities $q_j r_k$ and $\sum_j q_j r_j$ for the continuous CNLS equations (1.2).

It is also remarkable that higher than second conserved densities for (2.28) have expressions such as

$$\prod_{k=1}^{i-1}\left(1-\sum_j q_{n+k}^{(j)} r_{n+k}^{(j)}\right)\sum_j \left\{q_{n+i}^{(j)} r_n^{(j)} + (-1)^i q_n^{(j)} r_{n+i}^{(j)}\right\} + \cdots \qquad (i \geqslant 2)$$

which split into two independent densities with simpler structures,

$$\prod_{k=1}^{i-1}\left(1-q_{n+k}r_{n+k}\right) q_{n+i}r_n + \cdots \qquad (i \geqslant 2)$$

and

$$\prod_{k=1}^{i-1}\left(1-q_{n+k}r_{n+k}\right) q_n r_{n+i} + \cdots \qquad (i \geqslant 2)$$

in the single-component case (cf equation (1.4)). This fact implies that a direct recursion formula of the conserved densities for the sd-CNLS equations might be, if it exists, very complicated. We have obtained the first four conserved densities concisely by use of the recursion formula of the conserved densities for the sd-matrix NLS equation (2.6).

3. Inverse scattering method

In this section we investigate the scattering and inverse scattering problems associated with the $2l \times 2l$ ($l = 2^{m-1}$) matrix (2.3),

$$\begin{bmatrix} \Psi_{1\,n+1} \\ \Psi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z} F_1 & F_1 Q_n \\ F_2 R_n & F_2/\boldsymbol{z} \end{bmatrix} \begin{bmatrix} \Psi_{1\,n} \\ \Psi_{2\,n} \end{bmatrix} \tag{3.1}$$

to solve the sd-CNLS equations (2.30). Here and hereafter the superscripts (m) of $F_1^{(m)}$, $F_2^{(m)}$, $Q_n^{(m)}$ and $R_n^{(m)}$ are often omitted for convenience. To simplify the analysis, we consider a gauge transformation

$$\Phi_n = g_n \Psi_n \qquad g_n = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi n/4}(F_1)^n \mathrm{e}^{-\mathrm{i}H_1 t} & \\ & \mathrm{e}^{-\mathrm{i}\pi(n-1)/4}(F_2)^n \mathrm{e}^{-\mathrm{i}H_2 t} \end{bmatrix}. \tag{3.2}$$

Then the scattering problem (3.1) is changed into

$$\begin{bmatrix} \Phi_{1\,n+1} \\ \Phi_{2\,n+1} \end{bmatrix} = \begin{bmatrix} zI & \tilde{Q}_n \\ \tilde{R}_n & I/z \end{bmatrix} \begin{bmatrix} \Phi_{1\,n} \\ \Phi_{2\,n} \end{bmatrix} \tag{3.3}$$

where the transformed spectral parameter and potentials are

$$\begin{aligned} &z = \boldsymbol{z}\mathrm{e}^{\mathrm{i}\pi/4} \qquad \tilde{Q}_n = \mathrm{i}^n (F_1)^n\, \mathrm{e}^{-\mathrm{i}H_1 t} Q_n\, \mathrm{e}^{\mathrm{i}H_2 t}(F_2)^n \\ &\tilde{R}_n = (-\mathrm{i})^n (F_2)^n\, \mathrm{e}^{-\mathrm{i}H_2 t} R_n\, \mathrm{e}^{\mathrm{i}H_1 t}(F_1)^n. \end{aligned} \tag{3.4}$$

The constraints (2.32) and (2.31) lead to those for $\tilde{Q}_n$ and $\tilde{R}_n$,

$$\tilde{R}_n = -\tilde{Q}_n^\dagger \qquad \tilde{Q}_n \tilde{R}_n = \tilde{R}_n \tilde{Q}_n = -\sum_{j=1}^m |q_n^{(j)}|^2\, I \equiv -\sigma_n I. \tag{3.5}$$

We assume the rapidly decreasing boundary conditions,

$$\tilde{Q}_n, \tilde{R}_n \to O \qquad \text{as} \quad n \to \pm\infty. \tag{3.6}$$

Considering the time dependence of the scattering data, we can solve the initial-value problem of the sd-CNLS equations (2.30). Some of the main ideas in the following are an extension and a modification of the analyses for the matrix KdV equation [52], the matrix mKdV equations and the cmKdV equations [20] (see [48] for details).

3.1. Scattering problem

Let $\Phi_n^{(1)}(z)$ and $\Phi_n^{(2)}(z)$ be matrix solutions of (3.3) composed of $2l$ $(= 2^m)$ rows and l $(= 2^{m-1})$ columns. We introduce the following matrix function of $\Phi^{(1)}$ and $\Phi^{(2)}$:

$$W_n\left[\Phi^{(1)}, \Phi^{(2)}\right] \equiv \Phi_n^{(1)\dagger}\left(\frac{1}{z^*}\right)\Phi_n^{(2)}(z). \tag{3.7}$$

This satisfies a recursion relation

$$W_{n+1}\left[\Phi^{(1)}, \Phi^{(2)}\right] = \left(I - \tilde{Q}_n\tilde{R}_n\right)W_n\left[\Phi^{(1)}, \Phi^{(2)}\right] = \rho_n W_n\left[\Phi^{(1)}, \Phi^{(2)}\right] \tag{3.8}$$

where ρ_n is defined by

$$\rho_n \equiv 1 + \sigma_n = 1 + \sum_{j=1}^{m} |q_n^{(j)}|^2. \tag{3.9}$$

We introduce Jost functions ϕ_n, $\bar{\phi}_n$ and ψ_n, $\bar{\psi}_n$ which satisfy the boundary conditions

$$\phi_n \sim \begin{bmatrix} I \\ O \end{bmatrix} z^n \qquad \bar{\phi}_n \sim \begin{bmatrix} O \\ -I \end{bmatrix} z^{-n} \qquad \text{as} \quad n \to -\infty \tag{3.10}$$

and

$$\psi_n \sim \begin{bmatrix} O \\ I \end{bmatrix} z^{-n} \qquad \bar{\psi}_n \sim \begin{bmatrix} I \\ O \end{bmatrix} z^n \qquad \text{as} \quad n \to +\infty. \tag{3.11}$$

Here O and I are, respectively, the $l \times l$ zero matrix and the $l \times l$ unit matrix. We can show that $\phi_n z^{-n}$, $\psi_n z^n$ are analytic outside the unit circle ($|z| > 1$) on the complex z plane, and that $\bar{\phi}_n z^n$, $\bar{\psi}_n z^{-n}$ are analytic inside the unit circle ($|z| < 1$) on the z plane, when $\tilde{Q}_n$ and $\tilde{R}_n$ go to O sufficiently rapidly as $n \to \pm\infty$. We assume the following summation representation of the Jost functions ψ_n and $\bar{\psi}_n$:

$$\psi_n = \sum_{n'=n}^{\infty} z^{-n'} K(n, n') \qquad \bar{\psi}_n = \sum_{n'=n}^{\infty} z^{n'} \bar{K}(n, n') \tag{3.12}$$

where $K(n, n')$ and $\bar{K}(n, n')$ are z-independent column vectors which consist of two $l \times l$ square matrices,

$$K(n, n') = \begin{bmatrix} K_1(n, n') \\ K_2(n, n') \end{bmatrix} \qquad \bar{K}(n, n') = \begin{bmatrix} \bar{K}_1(n, n') \\ \bar{K}_2(n, n') \end{bmatrix}.$$

We substitute (3.12) into (3.3). Equating the terms with the same power of z, we obtain

$$K_1(n, n) = O \tag{3.13a}$$

$$K_2(n, n) = \prod_{i=n}^{\infty}(I - \tilde{R}_i\tilde{Q}_i)^{-1} = \prod_{i=n}^{\infty}\rho_i^{-1} I \tag{3.13b}$$

$$\tilde{Q}_n K_2(n, n) = -K_1(n, n+1) \tag{3.13c}$$

and

$$\bar{K}_2(n, n) = O \tag{3.14a}$$

$$\bar{K}_1(n, n) = \prod_{i=n}^{\infty}(I - \tilde{Q}_i\tilde{R}_i)^{-1} = \prod_{i=n}^{\infty}\rho_i^{-1} I \tag{3.14b}$$

$$\tilde{R}_n\bar{K}_1(n, n) = -\bar{K}_2(n, n+1). \tag{3.14c}$$

Since a pair of the Jost functions ϕ_n and $\bar{\phi}_n$, or ψ_n and $\bar{\psi}_n$, forms a fundamental system of the solutions of the scattering problem (3.3), we can set

$$\phi_n(z) = \bar{\psi}_n(z)\, A(z) + \psi_n(z)\, B(z) \tag{3.15a}$$

$$\bar{\phi}_n(z) = \bar{\psi}_n(z)\, \bar{B}(z) - \psi_n(z)\, \bar{A}(z). \tag{3.15b}$$

Here the coefficients $\{A(z),\ \bar{A}(z),\ B(z),\ \bar{B}(z)\}$ are n-independent $l \times l$ matrices which are called scattering data.

To derive the formula of the ISM rigorously and concisely, we assume that $\tilde{Q}_n$ and $\tilde{R}_n$ are on compact support. The result is, however, valid for larger classes of the potentials $\tilde{Q}_n$ and $\tilde{R}_n$. Using the asymptotic behaviours of the Jost functions (3.10), (3.11) and the relation (3.8), we obtain

$$A(z) = W_\infty[\bar{\psi}, \phi] \qquad \bar{A}(z) = -W_\infty[\psi, \bar{\phi}] \tag{3.16}$$

and

$$A^\dagger\left(\frac{1}{z^*}\right) A(z) + B^\dagger\left(\frac{1}{z^*}\right) B(z) = \bar{A}^\dagger\left(\frac{1}{z^*}\right) \bar{A}(z) + \bar{B}^\dagger\left(\frac{1}{z^*}\right) \bar{B}(z) = \prod_{n=-\infty}^{\infty} \rho_n\, I \tag{3.17a}$$

$$A^\dagger\left(\frac{1}{z^*}\right) \bar{B}(z) = B^\dagger\left(\frac{1}{z^*}\right) \bar{A}(z). \tag{3.17b}$$

The expressions (3.16) show that $A(z)$ and $\bar{A}(z)$ are, respectively, analytic outside the unit circle ($|z| > 1$) and inside the unit circle ($|z| < 1$).

3.2. Gel'fand–Levitan–Marchenko equations

Multiplying $A(z)^{-1}$ and $\bar{A}(z)^{-1}$ from the right to (3.15*a*) and (3.15*b*), respectively, we have

$$\phi_n(z)\, A(z)^{-1} = \bar{\psi}_n(z) + \psi_n(z)\, B(z)\, A(z)^{-1} \tag{3.18a}$$

$$\bar{\phi}_n(z)\, \bar{A}(z)^{-1} = -\psi_n(z) + \bar{\psi}_n(z)\, \bar{B}(z)\, \bar{A}(z)^{-1}. \tag{3.18b}$$

We substitute (3.12) into the right-hand side of (3.18*a*) and operate on both sides

$$\frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}z\, z^{-m-1} \qquad (m \geqslant n)$$

where C denotes a contour along the unit circle $|z| = 1$. It should be noticed that $\phi_n z^{-n}$ and $A(z)$ are analytic outside the unit circle C, $|z| > 1$. The inverse of $A(z)$, i.e. $A(z)^{-1}$, is given by

$$A(z)^{-1} = \frac{1}{\det A(z)}\, \tilde{A}(z)$$

where $\tilde{A}$ denotes the cofactor matrix of A. We assume that $1/\det A(z)$ is regular on the unit circle C and has $2N$ isolated simple poles $\{z_1, z_2, \ldots, z_{2N}\}$ in $|z| > 1$ (see (3.27) for the reason why we choose the number of poles to be $2N$). We set

$$J_{\infty,n} = \lim_{z\to\infty} \phi_n z^{-n} A(z)^{-1}$$

and use the residue theorem. After some computation, we arrive at the discrete version of the Gel'fand–Levitan–Marchenko equation,

$$\bar{K}(n, m) + \sum_{n'=n}^{\infty} K(n, n')\, F(n' + m) = J_{\infty,n}\, \delta_{n,m} \qquad (m \geqslant n). \tag{3.19}$$

Here $F(n'+m)$ is defined by

$$F(n'+m) \equiv \frac{1}{2\pi \mathrm{i}} \oint_C B(z)\, A(z)^{-1} z^{-(n'+m)-1} \mathrm{d}z + \sum_{j=1}^{2N} C_j z_j^{-(n'+m)-1} \tag{3.20}$$

where C_j is the residue matrix of $B(z)\, A(z)^{-1}$ at $z = z_j$.

Similarly, we operate

$$\frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}z\, z^{m-1} \qquad (m \geqslant n)$$

on both sides of (3.18*b*) substituting (3.12). As has been mentioned previously, $\bar{\phi}_n z^n$ and $\bar{A}(z)$ are analytic inside the unit circle C, $|z| < 1$. We assume that $1/\det \bar{A}(z)$ is regular on the unit circle C and has $2\bar{N}$ isolated simple poles $\{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2\bar{N}}\}$ in $|z| < 1$. We set

$$\bar{J}_{0,n} = \lim_{z\to 0} \bar{\phi}_n z^n \bar{A}(z)^{-1}$$

and use the residue theorem. Finally, we obtain the counterpart of the discrete Gel'fand–Levitan–Marchenko equation,

$$K(n,m) - \sum_{n'=n}^{\infty} \bar{K}(n,n')\, \bar{F}(n'+m) = -\bar{J}_{0,n}\delta_{n,m} \qquad (m \geqslant n). \tag{3.21}$$

Here $\bar{F}(n'+m)$ is defined by

$$\bar{F}(n'+m) \equiv \frac{1}{2\pi \mathrm{i}} \oint_C \bar{B}(z)\, \bar{A}(z)^{-1} z^{n'+m-1}\, \mathrm{d}z - \sum_{k=1}^{2\bar{N}} \bar{C}_k\, \bar{z}_k^{n'+m-1}. \tag{3.22}$$

The matrix $\bar{C}_k$ is the residue matrix of $\bar{B}(z)\, \bar{A}(z)^{-1}$ at $z = \bar{z}_k$.

From (3.13*b*) and (3.14*b*), it is natural to set

$$K(n,m) = \kappa(n,m) \prod_{i=n}^{\infty} (I - \tilde{R}_i \tilde{Q}_i)^{-1} = \kappa(n,m) \prod_{i=n}^{\infty} \rho_i^{-1} \qquad (m \geqslant n)$$

$$\bar{K}(n,m) = \bar{\kappa}(n,m) \prod_{i=n}^{\infty} (I - \tilde{Q}_i \tilde{R}_i)^{-1} = \bar{\kappa}(n,m) \prod_{i=n}^{\infty} \rho_i^{-1} \qquad (m \geqslant n).$$

Here $\kappa(n,m)$ and $\bar{\kappa}(n,m)$ are column vectors whose elements are $l \times l$ square matrices,

$$\kappa(n,m) = \begin{bmatrix} \kappa_1(n,m) \\ \kappa_2(n,m) \end{bmatrix} \qquad \bar{\kappa}(n,m) = \begin{bmatrix} \bar{\kappa}_1(n,m) \\ \bar{\kappa}_2(n,m) \end{bmatrix}.$$

In particular, $\kappa(n,n)$ and $\bar{\kappa}(n,n)$ are given by

$$\kappa(n,n) = \begin{bmatrix} O \\ I \end{bmatrix} \qquad \bar{\kappa}(n,n) = \begin{bmatrix} I \\ O \end{bmatrix}.$$

Due to (3.13) and (3.14), the potentials $\tilde{Q}_n$ and $\tilde{R}_n$ are given by

$$-\kappa_1(n,n+1) = \tilde{Q}_n \tag{3.23}$$

$$-\bar{\kappa}_2(n,n+1) = \tilde{R}_n. \tag{3.24}$$

In terms of κ and $\bar{\kappa}$, the Gel'fand–Levitan–Marchenko equations (3.19) and (3.21) for $m > n$ are rewritten as

$$\bar{\kappa}(n,m) + \begin{bmatrix} O \\ I \end{bmatrix} F(n+m) + \sum_{n'=n+1}^{\infty} \kappa(n,n')\, F(n'+m) = \begin{bmatrix} O \\ O \end{bmatrix} \qquad (m > n) \tag{3.25}$$

$$\kappa(n,m) - \begin{bmatrix} I \\ O \end{bmatrix} \bar{F}(n+m) - \sum_{n'=n+1}^{\infty} \bar{\kappa}(n,n')\, \bar{F}(n'+m) = \begin{bmatrix} O \\ O \end{bmatrix} \qquad (m > n). \tag{3.26}$$

It should be noted that the scattering problem (3.3) gives the symmetry properties of the scattering data. For instance, we have

$$\det A(z) = \det A(-z) \qquad \det \bar{A}(z) = \det \bar{A}(-z) \tag{3.27}$$

which means that the eigenvalues z_j, $\bar{z}_k$ should appear as 'positive–negative' pairs. Further, we have

$$B(z)\, A(z)^{-1} = -B(-z)\, A(-z)^{-1} \qquad \bar{B}(z)\, \bar{A}(z)^{-1} = -\bar{B}(-z)\, \bar{A}(-z)^{-1}. \tag{3.28}$$

Therefore, we can simplify the forms of F and $\bar{F}$ as

$$F(n+m) = \begin{cases} 2F_R(n+m) & m = n+2j-1 \\ O & m = n+2j \end{cases} \qquad j \geqslant 1$$

$$F_R(n) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} B(z)\, A(z)^{-1} z^{-n-1}\, \mathrm{d}z + \sum_{j=1}^{N} C_j z_j^{-n-1}$$

and

$$\bar{F}(n+m) = \begin{cases} 2\bar{F}_R(n+m) & m = n+2j-1 \\ O & m = n+2j \end{cases} \qquad j \geqslant 1$$

$$\bar{F}_R(n) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} \bar{B}(z)\, \bar{A}(z)^{-1} z^{n-1}\, \mathrm{d}z - \sum_{k=1}^{\bar{N}} \bar{C}_k\, \bar{z}_k^{n-1}.$$

Here C_R denotes the right-half portion of the unit circle contour C.

The symmetry properties of F and $\bar{F}$ give rise to those of κ and $\bar{\kappa}$. From (3.25) and (3.26), we obtain

$$\kappa_1(n,m) = \begin{cases} \kappa_{1R}(n,m) & m = n+2j-1 \\ O & m = n+2j \end{cases} \qquad j \geqslant 1 \tag{3.29a}$$

$$\bar{\kappa}_2(n,m) = \begin{cases} \bar{\kappa}_{2R}(n,m) & m = n+2j-1 \\ O & m = n+2j \end{cases} \qquad j \geqslant 1. \tag{3.29b}$$

Considering the above symmetry properties, we obtain the simplified Gel'fand–Levitan–Marchenko equations for κ_{1R} and $\bar{\kappa}_{2R}$,

$$\kappa_{1R}(n,m) = 2\bar{F}_R(n+m) - 4 \sum_{\substack{n'=n+2 \\ n'-n=\text{even}}}^{\infty} \sum_{\substack{n''=n+1 \\ n''-n=\text{odd}}}^{\infty} \kappa_{1R}(n,n'')\, F_R(n''+n')\, \bar{F}_R(n'+m)$$
$$(m > n,\ m-n = \text{odd}) \tag{3.30}$$

$$\bar{\kappa}_{2R}(n,m) = -2F_R(n+m) - 4 \sum_{\substack{n'=n+2 \\ n'-n=\text{even}}}^{\infty} \sum_{\substack{n''=n+1 \\ n''-n=\text{odd}}}^{\infty} \bar{\kappa}_{2R}(n,n'')\, \bar{F}_R(n''+n')\, F_R(n'+m).$$
$$(m > n,\ m-n = \text{odd}) \tag{3.31}$$

3.3. Time dependence of the scattering data

Under the rapidly decreasing boundary conditions (3.6), the asymptotic form of the Lax matrix $\tilde{M}_n$ for the sd-CNLS equations (2.30) after the gauge transformation (3.2) is given by

$$\tilde{M}_n = g_n M_n g_n^{-1} + g_{n,t}\, g_n^{-1} \to \begin{bmatrix} z^2 I & O \\ O & I/z^2 \end{bmatrix} \qquad \text{as} \quad n \to \pm\infty.$$

We define time-dependent Jost functions by

$$\phi_n^{(t)} \equiv \phi_n e^{z^2 t} \sim \begin{bmatrix} I \\ O \end{bmatrix} z^n e^{z^2 t} \qquad \bar{\phi}_n^{(t)} \equiv \bar{\phi}_n e^{t/z^2} \sim \begin{bmatrix} O \\ -I \end{bmatrix} z^{-n} e^{t/z^2} \qquad \text{as} \quad n \to -\infty.$$

From the relations

$$\phi_{n,t}^{(t)} = \tilde{M}_n \phi_n^{(t)} \qquad \bar{\phi}_{n,t}^{(t)} = \tilde{M}_n \bar{\phi}_n^{(t)}$$

we obtain

$$\phi_{n,t} = (\tilde{M}_n - z^2 I)\phi_n \qquad \bar{\phi}_{n,t} = \left(\tilde{M}_n - \frac{1}{z^2} I\right)\bar{\phi}_n. \tag{3.32}$$

We put the definitions of the time-dependent scattering data,

$$\phi_n(z) = \bar{\psi}_n(z)\, A(z,t) + \psi_n(z)\, B(z,t)$$
$$\bar{\phi}_n(z) = \bar{\psi}_n(z)\, \bar{B}(z,t) - \psi_n(z)\, \bar{A}(z,t)$$

into (3.32). Then taking the limit $n \to +\infty$, we obtain the time dependences of A, BA^{-1}, C_j and $\bar{A}$, $\bar{B}\bar{A}^{-1}$, $\bar{C}_k$. They are given by, respectively,

$$\begin{aligned} &A(z,t) = A(z,0) \\ &B(z,t)\, A(z,t)^{-1} = B(z,0)\, A(z,0)^{-1} e^{-(z^2 - 1/z^2)t} \\ &C_j(t) = C_j(0)\, e^{-(z_j^2 - 1/z_j^2)t} \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} &\bar{A}(z,t) = \bar{A}(z,0) \\ &\bar{B}(z,t)\, \bar{A}(z,t)^{-1} = \bar{B}(z,0)\, \bar{A}(z,0)^{-1} e^{(z^2 - 1/z^2)t} \\ &\bar{C}_k(t) = \bar{C}_k(0)\, e^{(\bar{z}_k^2 - 1/\bar{z}_k^2)t}. \end{aligned} \tag{3.35}$$

The above results give explicitly time-dependent forms of $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n.

3.4. Initial-value problem

Thanks to the constraints $\tilde{R}_n = -\tilde{Q}_n^\dagger$ and $\tilde{Q}_n \tilde{R}_n = \tilde{R}_n \tilde{Q}_n = -\sigma_n I$, we have some additional relations besides (3.27) and (3.28). The first additional relation is

$$\det \bar{A}(z) = \left\{ \det A\left(\frac{1}{z^*}\right)\right\}^* \tag{3.36}$$

which is proved in [48]. This relation restricts the numbers and the positions of the poles of $1/\det A(z)$ and $1/\det \bar{A}(z)$, i.e.

$$\bar{N} = N \qquad \bar{z}_k = \frac{1}{z_k^*}. \tag{3.37}$$

Due to (3.17*b*), we have the second additional relations,

$$\bar{B}(z)\, \bar{A}(z)^{-1} = \{B(z)\, A(z)^{-1}\}^\dagger \qquad (\text{on } |z| = 1) \tag{3.38a}$$

$$\bar{C}_k = -\frac{1}{z_k^{*2}} C_k^\dagger. \tag{3.38b}$$

The relations (3.37) and (3.38) give a relation between $\bar{F}_R(n,t)$ and $F_R(n,t)$,

$$\bar{F}_R(n,t) = F_R(n,t)^\dagger. \tag{3.39}$$

In order to make the ISM applicable to the sd-CNLS equations, we have to take account of the internal symmetries of the potentials $\tilde{Q}_n$ and $\tilde{R}_n$. According to (2.25), (2.26) and (3.4), the potentials $\tilde{Q}_n$ and $\tilde{R}_n$ are defined recursively by

$$\tilde{Q}_n^{(1)} = \mathrm{i}^n \mathrm{e}^{2\mathrm{i}t} q_n^{(1)} \qquad \tilde{R}_n^{(1)} = (-\mathrm{i})^n \mathrm{e}^{-2\mathrm{i}t} r_n^{(1)}$$

$$\tilde{Q}_n^{(m+1)} = \begin{bmatrix} \tilde{Q}_n^{(m)} & \mathrm{i}^n \mathrm{e}^{2\mathrm{i}t} q_n^{(m+1)} I_{2^{m-1}} \\ (-\mathrm{i})^n \mathrm{e}^{-2\mathrm{i}t} r_n^{(m+1)} I_{2^{m-1}} & -\tilde{R}_n^{(m)} \end{bmatrix}$$

$$\tilde{R}_n^{(m+1)} = \begin{bmatrix} \tilde{R}_n^{(m)} & \mathrm{i}^n \mathrm{e}^{2\mathrm{i}t} q_n^{(m+1)} I_{2^{m-1}} \\ (-\mathrm{i})^n \mathrm{e}^{-2\mathrm{i}t} r_n^{(m+1)} I_{2^{m-1}} & -\tilde{Q}_n^{(m)} \end{bmatrix}.$$

If we set

$$\begin{aligned} \mathrm{i}^n \mathrm{e}^{2\mathrm{i}t} q_n^{(j)} &= v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} \\ (-\mathrm{i})^n \mathrm{e}^{-2\mathrm{i}t} r_n^{(j)} &= -v_n^{(2j-2)} + \mathrm{i} v_n^{(2j-1)} \end{aligned} \qquad j = 1, 2, \ldots, m \tag{3.40}$$

$\tilde{Q}_n^{(m)}$ and $\tilde{R}_n^{(m)}$ for $m \geqslant 2$ are written as

$$\tilde{Q}_n^{(m)} = v_n^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)} e_k \qquad \tilde{R}_n^{(m)} = -v_n^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)} e_k. \tag{3.41}$$

Here $\mathbb{1}$ is the $2^{m-1} \times 2^{m-1}$ unit matrix, which has been denoted by $I_{2^{m-1}}$. Because of $r_n^{(j)} = -q_n^{(j)*}$, $v_n^{(0)}$ and $v_n^{(k)}$ should be real. Substitution of (3.41) into the relations (3.5) yields the following important relations for $2^{m-1} \times 2^{m-1}$ matrices $\{e_i\}$:

$$\{e_i, e_j\}_+ = -2\delta_{i,j} \mathbb{1} \tag{3.42}$$

$$e_k^\dagger = -e_k. \tag{3.43}$$

Here $\{\cdot, \cdot\}_+$ denotes the anti-commutator. It is stressed that expressions (3.41) with conditions (3.42) and (3.43) are only a sufficient (i.e. not necessary) condition of (3.5).

Considering the above symmetries of the potentials $\tilde{Q}_n^{(m)}$ and $\tilde{R}_n^{(m)}$ for $m \geqslant 2$, we can show the following properties of the scattering data.

Proposition 3.4.1. *(a) The determinant of $A(z)$ and the determinant of $A(z^*)$ are related by*

$$\det A(z) = \{\det A(z^*)\}^*. \tag{3.44}$$

Thus the poles of $1/\det A(z)$ in $|z| > 1$ appear as pairs situated symmetric with respect to the real axis. Therefore, we replace $2N$ in section 3.2 with $4N$ and choose the values of $2N$ poles in $|z| > 1$, $\mathrm{Re}\, z \geqslant 0$ as

$$\begin{aligned} z_{2j-1} &= \xi_j + \mathrm{i}\eta_j = a_j\, \mathrm{e}^{\mathrm{i}\theta_j} \\ z_{2j} &= z_{2j-1}^* = \xi_j - \mathrm{i}\eta_j = a_j\, \mathrm{e}^{-\mathrm{i}\theta_j} \end{aligned} \qquad j = 1, 2, \ldots, N \tag{3.45}$$

where $a_j > 1$, $0 < \theta_j \leqslant \pi/2$ for $\theta_j \neq 0$. The conditions (3.45) should be interpreted as follows. If $\theta_j = 0$, the corresponding pole does not need its counterpart. The values of the remaining $2N$ poles in $|z| > 1$, $\mathrm{Re}\, z \leqslant 0$ are given by

$$z_{2N+k} = -z_k \qquad k = 1, 2, \ldots, 2N.$$

(b) The reflection coefficient $B(z)A(z)^{-1}$ is expressed as

$$B(z)\, A(z)^{-1} = r^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} r^{(k)} e_k. \tag{3.46}$$

Here $r^{(0)}$ and $r^{(k)}$ are complex functions of z and t which satisfy

$$r^{(0)}(z^*) = r^{(0)}(z)^* \qquad r^{(k)}(z^*) = r^{(k)}(z)^*. \tag{3.47}$$

(c) *The residue matrices* $\{C_1, C_2, \ldots, C_{2N}\}$ *are expressed as*

$$\begin{aligned} C_{2j-1} &= c_j^{(0)} \mathbb{1} + \sum_{k=1}^{2m-1} c_j^{(k)} e_k \\ C_{2j} &= c_j^{(0)*} \mathbb{1} + \sum_{k=1}^{2m-1} c_j^{(k)*} e_k \end{aligned} \qquad j = 1, 2, \ldots, N \tag{3.48}$$

with complex functions of time t, $c_j^{(0)}$ *and* $c_j^{(k)}$.

The statements (a)–(c) are proved essentially in the same way as in [48] and therefore their proofs are omitted.

Taking account of the above conditions, we obtain explicit expressions of $F_R(n, t)$ and $\bar{F}_R(n, t)$ for odd n,

$$\begin{aligned} F_R(n,t) &= \frac{1}{2\pi \mathrm{i}} \int_{C_R} B(z)\, A(z)^{-1} z^{-n-1} \mathrm{d}z + \sum_{j=1}^{2N} C_j z_j^{-n-1} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{C_{UR}} \left\{ \left(r^{(0)} z^{-n-1} + r^{(0)*} z^{n-1} \right) \mathbb{1} + \sum_{k=1}^{2m-1} \left(r^{(k)} z^{-n-1} + r^{(k)*} z^{n-1} \right) e_k \right\} \mathrm{d}z \\ &\quad + \sum_{j=1}^{N} \left\{ \left(c_j^{(0)} z_j^{-n-1} + c_j^{(0)*} z_j^{*\,-n-1} \right) \mathbb{1} + \sum_{k=1}^{2m-1} \left(c_j^{(k)} z_j^{-n-1} + c_j^{(k)*} z_j^{*\,-n-1} \right) e_k \right\} \end{aligned} \tag{3.49}$$

$$\begin{aligned} \bar{F}_R(n,t) &= F_R(n,t)^\dagger \\ &= \frac{1}{2\pi \mathrm{i}} \int_{C_{UR}} \left\{ \left(r^{(0)} z^{-n-1} + r^{(0)*} z^{n-1} \right) \mathbb{1} - \sum_{k=1}^{2m-1} \left(r^{(k)} z^{-n-1} + r^{(k)*} z^{n-1} \right) e_k \right\} \mathrm{d}z \\ &\quad + \sum_{j=1}^{N} \left\{ \left(c_j^{(0)} z_j^{-n-1} + c_j^{(0)*} z_j^{*\,-n-1} \right) \mathbb{1} - \sum_{k=1}^{2m-1} \left(c_j^{(k)} z_j^{-n-1} + c_j^{(k)*} z_j^{*\,-n-1} \right) e_k \right\} \end{aligned} \tag{3.50}$$

where C_{UR} denotes a contour along the quadrant (upper-right portion) of the unit circle C. We see that the coefficients of $\mathbb{1}$ and $\{e_k\}$ in (3.49) and (3.50) are real. Thus, a pair of $\bar{F}_R$ and $-F_R$ is expressed in the same form as (3.41), as is expected from the viewpoint of successive approximations for the Gel'fand–Levitan–Marchenko equations.

Because $B(z)A(z)^{-1}$ and C_j depend on t as (3.34), the time dependences of $r^{(0)}$, $r^{(k)}$ and $c_j^{(0)}$, $c_j^{(k)}$ are given by

$$r^{(0)}(z,t) = r^{(0)}(z,0)\, \mathrm{e}^{-(z^2-1/z^2)t} \qquad r^{(k)}(z,t) = r^{(k)}(z,0)\, \mathrm{e}^{-(z^2-1/z^2)t} \tag{3.51}$$

$$c_j^{(0)}(t) = c_j^{(0)}(0)\, \mathrm{e}^{-(z_j^2-1/z_j^2)t} \qquad c_j^{(k)}(t) = c_j^{(k)}(0)\, \mathrm{e}^{-(z_j^2-1/z_j^2)t}. \tag{3.52}$$

Combining (3.30) and (3.31) with (3.39), we arrive at

$$\begin{aligned} \kappa_{1R}(n,m;t) &= 2F_R(n+m,t)^\dagger - 4 \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \kappa_{1R}(n, n+2l_2-1; t)\, F_R(2n+2l_2+2l_1-1, t) \\ &\qquad \times F_R(n+2l_1+m, t)^\dagger \end{aligned} \tag{3.53}$$

$$\begin{aligned} \bar{\kappa}_{2R}(n,m;t) &= -2F_R(n+m,t) - 4 \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \bar{\kappa}_{2R}(n, n+2l_2-1; t)\, F_R(2n+2l_2+2l_1-1, t)^\dagger \\ &\qquad \times F_R(n+2l_1+m, t) \end{aligned} \tag{3.54}$$

for $m > n$, $m - n =$ odd, where $F_R(n, t)$ is given by (3.49).

Now the initial-value problem of the sd-CNLS equations (2.30) can be solved in the following steps.

(a) For given potentials at $t = 0$, $\tilde{Q}_n(0)$ and $\tilde{R}_n(0)$ which are expressed as (3.41), we solve the scattering problem (3.3), and obtain the scattering data $\{B(z)A(z)^{-1}, z_j, C_j\}$ or, more concretely $\{r^{(0)}, r^{(k)}, z_j, c_j^{(0)}, c_j^{(k)}\}$.
(b) The time dependence of the scattering data is given by (3.34) (or (3.51) and (3.52)).
(c) We substitute the time dependence of the scattering data into the Gel'fand–Levitan–Marchenko equations (3.53) and (3.54). Solving the equations, we reconstruct the time-dependent potentials,

$$\tilde{Q}_n(t) = -\kappa_{1R}(n, n+1; t) \qquad \tilde{R}_n(t) = -\bar{\kappa}_{2R}(n, n+1; t).$$

This solution proves directly the complete integrability of the sd-CNLS equations (2.30).

3.5. Soliton solutions

To construct soliton solutions of the sd-CNLS equations, we assume the reflection-free condition, i.e. $B(z) = \bar{B}(z) = O$ on $|z| = 1$. Then, $F_R(n,t)$ and $\bar{F}_R(n,t)$ for odd n are given by

$$F_R(n,t) = \sum_{j=1}^{2N} C_j(t)\, z_j^{-n-1} \qquad C_j(t) = C_j(0)\, \mathrm{e}^{-(z_j^2 - 1/z_j^2)t} \tag{3.55}$$

$$\bar{F}_R(n,t) = -\sum_{k=1}^{2N} \bar{C}_k(t)\, \bar{z}_k^{n-1} \qquad \bar{C}_k(t) = \bar{C}_k(0)\, \mathrm{e}^{(\bar{z}_k^2 - 1/\bar{z}_k^2)t}. \tag{3.56}$$

To solve (3.30) (or equation (3.53)) with (3.55) and (3.56), we set

$$\kappa_{1R}(n,m;t) = \sum_{k=1}^{2N} P_k \bar{C}_k(t)\, \bar{z}_k^{n+m-1} \qquad (m-n = \text{odd}). \tag{3.57}$$

Substituting (3.57) into (3.30) or (3.53), we have

$$P_k - 4\sum_{l=1}^{2N}\sum_{j=1}^{2N} \left(\frac{\bar{z}_l}{z_j}\right)^{2n} \frac{\bar{z}_k^2}{\left(z_j^2 - \bar{z}_k^2\right)\left(z_j^2 - \bar{z}_l^2\right)} P_l \bar{C}_l(t)\, C_j(t) = -2I. \tag{3.58}$$

In terms of a matrix S whose elements are defined by

$$\begin{aligned} S_{lk} &\equiv \delta_{l,k} I - 4\sum_{j=1}^{2N} \left(\frac{\bar{z}_l}{z_j}\right)^{2n} \frac{\bar{z}_k^2}{\left(z_j^2 - \bar{z}_k^2\right)\left(z_j^2 - \bar{z}_l^2\right)} \bar{C}_l(t)\, C_j(t) \\ &= \delta_{l,k} I + 4\sum_{j=1}^{2N} \frac{1}{z_j^{2n} z_l^{*2n} \left(z_j^2 z_k^{*2} - 1\right)\left(z_j^2 z_l^{*2} - 1\right)} C_l(t)^\dagger C_j(t) \qquad 1 \leqslant l,k \leqslant 2N \end{aligned}$$

equation (3.58) is expressed by

$$(P_1\ P_2\ \cdots\ P_{2N}) \begin{pmatrix} S_{11} & \cdots & S_{1\,2N} \\ \vdots & \ddots & \vdots \\ S_{2N\,1} & \cdots & S_{2N\,2N} \end{pmatrix} = -2(\underbrace{I\ I\ \cdots\ I}_{2N}). \tag{3.59}$$

Similarly, we solve (3.31) (or equation (3.54)) with (3.55) and (3.56). Substitution of

$$\bar{\kappa}_{2R}(n,m;t) = \sum_{j=1}^{2N} \bar{P}_j C_j(t)\, z_j^{-(n+m)-1} \qquad (m-n = \text{odd}) \tag{3.60}$$

into (3.31) or (3.54) gives

$$\bar{P}_j - 4\sum_{l=1}^{2N}\sum_{k=1}^{2N}\frac{\bar{z}_k^{2n+2}}{z_l^{2n}}\frac{1}{\left(z_l^2-\bar{z}_k^2\right)\left(z_j^2-\bar{z}_k^2\right)}\bar{P}_l C_l(t)\,\bar{C}_k(t) = -2I. \tag{3.61}$$

Using a matrix $\bar{S}$,

$$\begin{aligned}\bar{S}_{lk} &\equiv \delta_{l,k}I - 4\sum_{j=1}^{2N}\frac{\bar{z}_j^{2n+2}}{z_l^{2n}}\frac{1}{\left(z_l^2-\bar{z}_j^2\right)\left(z_k^2-\bar{z}_j^2\right)}C_l(t)\,\bar{C}_j(t)\\ &= \delta_{l,k}I + 4\sum_{j=1}^{2N}\frac{1}{z_l^{2n}z_j^{*\,2n}\left(z_l^2 z_j^{*\,2}-1\right)\left(z_k^2 z_j^{*\,2}-1\right)}C_l(t)\,C_j(t)^\dagger \qquad 1\leqslant l,k\leqslant 2N\end{aligned}$$

we rewrite (3.61) as

$$(\bar{P}_1\bar{P}_2\cdots\bar{P}_{2N})\begin{pmatrix}\bar{S}_{11} & \cdots & \bar{S}_{1\,2N}\\ \vdots & \ddots & \vdots\\ \bar{S}_{2N\,1} & \cdots & \bar{S}_{2N\,2N}\end{pmatrix} = -2(\underbrace{I\ I\ \cdots\ I}_{2N}). \tag{3.62}$$

Equations (3.59) and (3.62) are readily solved. Thus the N-soliton solution of the sd-CNLS equations (2.30) is given by

$$\begin{aligned}\tilde{Q}_n^{(m)}(t) &= \mathrm{i}^n(F_1)^n\,\mathrm{e}^{-\mathrm{i}H_1t}\,Q_n^{(m)}\,\mathrm{e}^{\mathrm{i}H_2t}(F_2)^n = -\kappa_{1R}(n,n+1;t)\\ &= -2\,(\underbrace{I\ I\ \cdots\ I}_{2N})\ S^{-1}\begin{pmatrix}C_1(t)^\dagger/z_1^{*\,2n+2}\\ C_2(t)^\dagger/z_2^{*\,2n+2}\\ \vdots\\ C_{2N}(t)^\dagger/z_{2N}^{*\,2n+2}\end{pmatrix}\end{aligned} \tag{3.63a}$$

$$\begin{aligned}\tilde{R}_n^{(m)}(t) &= (-\mathrm{i})^n(F_2)^n\,\mathrm{e}^{-\mathrm{i}H_2t}\,R_n^{(m)}\,\mathrm{e}^{\mathrm{i}H_1t}(F_1)^n = -\bar{\kappa}_{2R}(n,n+1;t)\\ &= 2\,(\underbrace{I\ I\ \cdots\ I}_{2N})\ \bar{S}^{-1}\begin{pmatrix}C_1(t)/z_1^{2n+2}\\ C_2(t)/z_2^{2n+2}\\ \vdots\\ C_{2N}(t)/z_{2N}^{2n+2}\end{pmatrix}.\end{aligned} \tag{3.63b}$$

Strictly speaking, equation (3.63) includes breathers besides solitons. In order to extract pure soliton solutions, we assume that each soliton seen in $\sum_j |q_n^{(j)}(t)|^2$ has a time-independent shape. By calculating an asymptotic behaviour of the tails of solitons at $n\to+\infty$, we obtain the corresponding necessary conditions

$$C_{2j-1}\bar{C}_{2j} = \bar{C}_{2j}C_{2j-1} = C_{2j}\bar{C}_{2j-1} = \bar{C}_{2j-1}C_{2j} = O \qquad j=1,2,\ldots,N \tag{3.64}$$

on the residue matrices. Equation (3.64) is translated explicitly into

$$\sum_{i=0}^{2m-1}\left(c_j^{(i)}\right)^2 = 0 \qquad j=1,2,\ldots,N. \tag{3.65}$$

As an example, we write down a pure one-soliton solution. Choose $N = 1$ and set

$$\bar{z}_1 = \frac{1}{z_1^*} = \mathrm{e}^{-W+\mathrm{i}\theta} \qquad W > 0$$
$$\bar{C}_1 = -\frac{1}{z_1^{*\,2}} C_1^\dagger = -\frac{1}{z_1^{*\,2}}\left(c_1^{(0)\,*}\mathbb{1} - \sum_{k=1}^{2m-1} c_1^{(k)\,*} e_k \right) \equiv \bar{c}_1^{(0)}\mathbb{1} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)} e_k$$
$$\bar{C}_2 = \bar{c}_1^{(0)\,*}\mathbb{1} + \sum_{k=1}^{2m-1} \bar{c}_1^{(k)\,*} e_k$$
$$\mathrm{e}^{\phi_0} = \frac{\sinh 2W}{\sqrt{2\sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}}. \tag{3.66}$$

Then, from (3.63) we obtain

$$\tilde{Q}_n^{(m)}(t) = \mathrm{sech}\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\} \frac{\sinh 2W}{\sqrt{2\sum_{j=0}^{2m-1} |\bar{c}_1^{(j)}(0)|^2}}$$
$$\times \{\bar{C}_1(0)\,\mathrm{e}^{2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}} + \bar{C}_2(0)\,\mathrm{e}^{-2\mathrm{i}\{n\theta + (\cosh 2W \sin 2\theta)t\}}\} \tag{3.67a}$$
$$\tilde{R}_n^{(m)}(t) = -\tilde{Q}_n^{(m)}(t)^\dagger. \tag{3.67b}$$

It is straightforward to show that (3.67) can be expressed as (3.41) with real coefficients $v_n^{(i)}(t)$ of $\mathbb{1}$ and $\{e_k\}$. Thus we have checked in terms of the inverse problem that the conditions (3.41) or consequently (3.5) are satisfied under proposition 3.4.1 and the conditions (3.64), in the case of the one-soliton solution.

By introducing a new set of constants by

$$\alpha_i \equiv \bar{c}_1^{(2i-2)}(0) + \mathrm{i}\bar{c}_1^{(2i-1)}(0) \qquad i = 1, 2, \ldots, m$$
$$\beta_i \equiv \bar{c}_1^{(2i-2)}(0) - \mathrm{i}\bar{c}_1^{(2i-1)}(0)$$

we obtain a simplified expression of the pure one-soliton solution of the sd-CNLS equations (2.30), namely,

$$q_n^{(i)}(t) = \mathrm{sech}\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\} \frac{\sinh 2W}{\sqrt{\sum_{j=1}^{m}(|\alpha_j|^2 + |\beta_j|^2)}}$$
$$\times [\alpha_i\,\mathrm{e}^{2\mathrm{i}\{n(\theta-\pi/4)+(\cosh 2W \sin 2\theta - 1)t\}} + \beta_i^*\,\mathrm{e}^{-2\mathrm{i}\{n(\theta+\pi/4)+(\cosh 2W \sin 2\theta+1)t\}}]$$
$$i = 1, 2, \ldots, m. \tag{3.68}$$

Here the condition (3.65) for $N = 1$ is cast into the orthogonality condition,

$$\sum_{i=1}^{m} \alpha_i \beta_i = 0.$$

The soliton solution (3.68) exhibits a novel property as a solution of NLS-type equations. Because there are two carrier waves in one envelope soliton, the shape of soliton observed in $|q_n^{(i)}(t)|^2$ periodically oscillates in time. It is observed for (3.68) that the summation of $|q_n^{(i)}(t)|^2$ with respect to components, $i\ (= 1, 2, \ldots, m)$

$$\sum_{i=1}^{m} |q_n^{(i)}(t)|^2 = \sinh^2 2W\,\mathrm{sech}^2\{2nW + 2(\sinh 2W \cos 2\theta)t + \phi_0\}$$

has a time-independent shape, as is expected. This fact suggests that the conditions (3.64) are necessary and sufficient to give pure soliton solutions even in the N-soliton case.

The continuum limit which reduces (3.68) into the one-soliton solution of the continuous CNLS equations can be seen as follows. We denote by h the (dimensionless) lattice spacing of the sd-CNLS model. We rescale t by

$$t \to \frac{1}{h^2} t$$

and set

$$q_n^{(i)}(t) = h u_i(x,t) \qquad x = nh$$
$$W = h\eta \qquad \theta - \frac{\pi}{4} = -h\xi$$
$$\alpha_i = h\gamma_i \qquad \beta_i = 0.$$

If we take the continuum limit $h \to 0$, the one-soliton solution (3.68) with (3.66) is transformed into

$$u_i(x,t) = 2\eta\, \mathrm{sech}\{2\eta x + 8\eta\xi t + \phi_0\} \frac{\gamma_i}{\sqrt{\sum_{j=1}^{m} |\gamma_j|^2}}\, \mathrm{e}^{-2\mathrm{i}\xi x - 4\mathrm{i}(\xi^2 - \eta^2)t} \qquad i = 1, 2, \ldots, m$$

with

$$\mathrm{e}^{\phi_0} = \frac{2\eta}{\sqrt{\sum_{j=1}^{m} |\gamma_j|^2}}.$$

This is indeed the one-soliton solution of the continuous CNLS equations,

$$\mathrm{i}\frac{\partial u_i}{\partial t} + \frac{\partial^2 u_i}{\partial x^2} + 2\sum_{j=1}^{m} |u_j|^2\, u_i = 0 \qquad i = 1, 2, \ldots, m. \tag{3.69}$$

It is noteworthy that either $\alpha_i = 0$ $(i = 1, 2, \ldots, m)$ or $\beta_i = 0$ $(i = 1, 2, \ldots, m)$ is necessary for us to take the continuum limit. This reflects the fact that the pure N-soliton solution of the sd-CNLS equations (2.30) includes more arbitrary constants than the N-soliton solution of the continuous CNLS equations (3.69). It has its origin in the oscillation of solitons seen in each component $|q_n^{(i)}(t)|^2$. In this sense, the structure of the pure soliton solution may be more similar to that of the cmKdV equations (1.3) rather than to that of the CNLS equations (3.69).

Ohta [46] obtained an N-soliton solution for the sd-CNLS equations in the Pfaffian representations. The N-soliton solution (3.63) with the constraints (3.64) contains more parameters than Ohta's N-soliton solution. Therefore, it is reasonable to conjecture that our solution reduces to Ohta's by a particular choice of those parameters.

We wish to show that (3.63*a*) and (3.63*b*) are expressed as

$$\tilde{Q}_n^{(m)}(t) = v_n^{(0)}(t)\mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)}(t)\, e_k \tag{3.70a}$$

$$\tilde{R}_n^{(m)}(t) = -v_n^{(0)}(t)\mathbb{1} + \sum_{k=1}^{2m-1} v_n^{(k)}(t)\, e_k \tag{3.70b}$$

without using the products of $\{e_i\}$ such as $e_i e_j, e_i e_j e_k$. Further, we expect $v_n^{(0)}, v_n^{(k)}$ to be real in (3.70*a*) and (3.70*b*). As mentioned previously, '(3.70) → (3.5)' holds. Thus, equation (3.5) is automatically satisfied if we can show (3.70). So far we have proved either (3.70*a*) or (3.70*b*) only for $m = 2$. However, the result for the one-soliton solution implies that both of (3.70) are simultaneously satisfied under proposition 3.4.1.

4. Discussions

We have investigated the semi-discrete coupled nonlinear Schrödinger (sd-CNLS) equations. The analysis from the ISM point of view is given for the first time in this paper. A previous paper [48] dealt with a new extension of the semi-discrete version of the ISM proposed by Ablowitz and Ladik to solve the semi-discrete coupled modified KdV (sd-cmKdV) equations (1.6). In this paper, we have developed the extension with the help of the transformation (1.7) to solve the model considered with arbitrarily multiple components.

We should comment on the Lax formulations and the ISM for both the sd-cmKdV equations and the sd-CNLS equations, which may spotlight mysterious structure of discrete soliton equations.

First, there is an essential difference between the form of the L_n-matrix for the sd-CNLS equations and that for the sd-cmKdV equations in the case of $m \geqslant 2$ ($M \geqslant 4$). It seems that the scattering problem for the sd-CNLS equations associated with the L_n-matrix, e.g. equation (2.29), does not agree with the scattering problem for the CNLS equations [8, 53] in the continuum limit ($z = \mathrm{e}^{-\mathrm{i}\zeta h + \mathrm{i}\varphi}$, $h \to 0$). The situation is not observed for the continuum limit of the sd-cmKdV equations. This difference can be understood by considering the continuum limit of the semi-discrete coupled Hirota equations. A detailed explanation will be reported in a subsequent paper.

Secondly, it is obvious that (2.6) does not generally allow us to assume the reduction $R_n = \pm Q_n^\dagger$, because of the order of the products. To consider the reduction $R_n = \pm Q_n^\dagger$, we should impose the additional restriction $Q_n R_n = R_n Q_n =$ scalar on Q_n and R_n. Thus, both (2.31) and (2.32) play a crucial role in our theory, which is peculiar to the discrete theory with multiple components.

Thirdly, as for the consistency of the N-soliton solution, it is difficult to prove both of equation (3.70) from the restrictions on the scattering data (3.45)–(3.48). In the continuous theory [20], (3.70) is easily proved at least for 2×2 matrices $Q^{(2)}$ and $R^{(2)}$. However, even in this case, it is not so easy to prove both of (3.70) in the discrete theory.

Fourthly, due to the relation (3.44), we always have to consider pairs of poles as scattering data in the ISM. The constraint gives the novel structure of solutions, 'two carrier waves in one envelope soliton'. This situation is closely related to the high internal symmetries of $Q_n^{(m)}$ and $R_n^{(m)}$ which may give a rich variety of dynamical behaviours in the multi-field soliton systems. As an influence of this fact, we need to assume additional conditions (3.64) to exclude breather-type solutions.

The sd-CNLS equations can be cast into alternative expressions by some transformations. We set a pair of variables $q_n^{(j)}$ and $r_n^{(j)}$ to be constant in (1.5), for instance, $q_n^{(m)} r_n^{(m)} = 1$ and consider a transformation of variables,

$$\begin{aligned} q_n^{(j)} \mathrm{e}^{2\mathrm{i}t} &\to q_n^{(j)} \\ r_n^{(j)} \mathrm{e}^{-2\mathrm{i}t} &\to r_n^{(j)} \end{aligned} \qquad j = 1, 2, \ldots, m-1.$$

Then, we obtain a simplified deformation of (1.5),

$$\begin{aligned} \mathrm{i}\frac{\partial q_n^{(j)}}{\partial t} &= \sum_{k=1}^{m-1} q_n^{(k)} r_n^{(k)} \left(q_{n+1}^{(j)} + q_{n-1}^{(j)} \right) \\ \mathrm{i}\frac{\partial r_n^{(j)}}{\partial t} &= -\sum_{k=1}^{m-1} r_n^{(k)} q_n^{(k)} \left(r_{n+1}^{(j)} + r_{n-1}^{(j)} \right) \end{aligned} \qquad j = 1, 2, \ldots, m-1.$$

To consider a multi-field extension of the full-discrete NLS equation [5, 6, 44] by our method, we need to modify the L_n-matrix (2.3) appropriately. The details of the analysis will be reported in a separate paper.

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Appendix. Trace formulae

In this appendix, we show interrelations between the generator of the conserved densities $\mathrm{tr}\{\log(I + Q_n\Gamma_n/z)\}$ for (2.6) in section 2.2 and the scattering data in section 3.

Let us recall that we have transformed the scattering problem (3.1) into the scattering problem (3.3) by the gauge transformation (3.2). For the moment, we do not impose restrictions such as (2.31) and (2.32) (or equation (3.5)) on square matrices Q_n and R_n and consider the sd-matrix NLS equation (2.6). The conditions (2.5) are assumed. We supplement some definitions and relations. First, we define the inverse of (3.15) by

$$\bar{\psi}_n(z) = \phi_n(z)\,\mathcal{A}(z) + \bar{\phi}_n(z)\,\mathcal{B}(z)$$
$$\psi_n(z) = \phi_n(z)\,\bar{\mathcal{B}}(z) - \bar{\phi}_n(z)\,\bar{\mathcal{A}}(z).$$

Secondly, the generator of the conserved densities is invariant under the gauge transformation (3.2):

$$\begin{aligned}\mathrm{tr}\log\left(I + \frac{1}{z}\tilde{Q}_n\Phi_{2n}\Phi_{1n}^{-1}\right) &= \mathrm{tr}\log\left\{I + \frac{1}{z\mathrm{e}^{\mathrm{i}\pi/4}}\,\mathrm{e}^{\mathrm{i}\pi n/2}(F_1)^n\,\mathrm{e}^{-\mathrm{i}H_1t}\,Q_n\,\mathrm{e}^{\mathrm{i}H_2t}(F_2)^n\right.\\ &\quad\left.\times\mathrm{e}^{-\mathrm{i}\pi(n-1)/4}(F_2)^n\,\mathrm{e}^{-\mathrm{i}H_2t}\,\Psi_{2n}\,\Psi_{1n}^{-1}\,\mathrm{e}^{\mathrm{i}H_1t}(F_1)^n\,\mathrm{e}^{-\mathrm{i}\pi n/4}\right\}\\ &= \mathrm{tr}\log\left(I + \frac{1}{z}Q_n\Psi_{2n}\Psi_{1n}^{-1}\right).\end{aligned}\qquad\text{(A.1)}$$

Thirdly, the asymptotic behaviours of the Jost functions ϕ_n and $\bar{\psi}_n$ are given by

$$\begin{aligned}\phi_n \equiv \begin{bmatrix}\phi_{1n}\\ \phi_{2n}\end{bmatrix} &\sim \begin{bmatrix}I\\ O\end{bmatrix}z^n &&\text{as}\quad n\to-\infty\\ &\sim \begin{bmatrix}A(z)\,z^n\\ B(z)\,z^{-n}\end{bmatrix} &&\text{as}\quad n\to+\infty\\ \bar{\psi}_n \equiv \begin{bmatrix}\bar{\psi}_{1n}\\ \bar{\psi}_{2n}\end{bmatrix} &\sim \begin{bmatrix}\mathcal{A}(z)\,z^n\\ -\mathcal{B}(z)\,z^{-n}\end{bmatrix} &&\text{as}\quad n\to-\infty\\ &\sim \begin{bmatrix}I\\ O\end{bmatrix}z^n &&\text{as}\quad n\to+\infty.\end{aligned}$$

Further, we can prove that $\phi_{2n}\phi_{1n}^{-1}$ is a polynomial in $1/z$ and $\bar{\psi}_{2n}\bar{\psi}_{1n}^{-1}$ is a polynomial in z. Therefore, we can replace $\phi_{2n}\phi_{1n}^{-1}$ and $\bar{\psi}_{2n}\bar{\psi}_{1n}^{-1}$ by $\Gamma_n^{(-)}$ and $\Gamma_n^{(+)}$ in section 2.2 respectively, except for the difference in gauge (see equation (3.2)). It is important that the ratios of two components, $\phi_{2n}\phi_{1n}^{-1}$ and $\bar{\psi}_{2n}\bar{\psi}_{1n}^{-1}$, are invariant when we consider the time-dependent Jost functions $\phi_n^{(t)} \equiv \phi_n \mathrm{e}^{z^2 t}$ and $\bar{\psi}_n^{(t)} \equiv \bar{\psi}_n \mathrm{e}^{z^2 t}$.

Now, we can relate the scattering data with the generator of the conserved densities. The determinant of $A(z)$, det $A(z)$, is expressed as

$$
\begin{aligned}
\log\det A(z) &= \operatorname{tr}\log A(z) = \operatorname{tr}\sum_{n=-\infty}^{\infty}\left[\log(\phi_{1\,n+1}z^{-n-1}) - \log(\phi_{1\,n}z^{-n})\right] \\
&= \operatorname{tr}\sum_{n=-\infty}^{\infty}\log\left[\phi_{1\,n+1}\phi_{1\,n}^{-1}z^{-1}\right] \\
&= \operatorname{tr}\sum_{n=-\infty}^{\infty}\log\left[I + \frac{1}{z}\tilde{Q}_n\phi_{2\,n}\phi_{1\,n}^{-1}\right] \\
&= \operatorname{tr}\sum_{n=-\infty}^{\infty}\log\left[I + \frac{1}{z}Q_n\Gamma_n^{(-)}\right] \\
&= \operatorname{tr}\sum_{n=-\infty}^{\infty}\bigg[\frac{1}{z^2}Q_nF_2R_{n-1}F_1 + \frac{1}{z^4}\Big\{Q_nR_{n-2} \\
&\quad - Q_nF_2R_{n-1}Q_{n-1}F_2R_{n-2} - \tfrac{1}{2}(Q_nF_2R_{n-1}F_1)^2\Big\} + \cdots\bigg].
\end{aligned}
\tag{A.2}
$$

Similarly, det $\mathcal{A}(z)$ is rewritten as

$$
\begin{aligned}
\log\det \mathcal{A}(z) &= -\operatorname{tr}\sum_{n=-\infty}^{\infty}\left[\log(\bar{\psi}_{1\,n+1}z^{-n-1}) - \log(\bar{\psi}_{1\,n}z^{-n})\right] \\
&= -\operatorname{tr}\sum_{n=-\infty}^{\infty}\log\left[I + \frac{1}{z}\tilde{Q}_n\bar{\psi}_{2\,n}\bar{\psi}_{1\,n}^{-1}\right] \\
&= -\operatorname{tr}\sum_{n=-\infty}^{\infty}\log\left[I + \frac{1}{z}Q_n\Gamma_n^{(+)}\right] \\
&= \operatorname{tr}\sum_{n=-\infty}^{\infty}\Big[-\log(I - Q_nR_n) + z^2Q_nF_2R_{n+1}F_1 \\
&\quad + z^4\Big\{Q_nR_{n+2} - Q_nR_{n+2}F_1Q_{n+1}R_{n+1}F_1 - \tfrac{1}{2}(Q_nF_2R_{n+1}F_1)^2\Big\} + \cdots\Big].
\end{aligned}
\tag{A.3}
$$

Here the time independence of $\mathcal{A}(z)$, $\mathcal{A}_t(z) = O$, is proved in the same manner as in section 3.3. It is now clear how the scattering data are expressed in terms of the integrals of motion for the sd-matrix NLS equation. However, it should be stressed that the above expansions do not yield local conservation laws. The method presented in section 2.2 is useful because it gives not only the densities but also the corresponding fluxes.

Conversely, the integrals of motion can be expressed in terms of the scattering data. For simplicity, we assume that

(a) $\tilde{Q}_n$ and $\tilde{R}_n$ are expressed as (3.41). Thus, proposition 3.4.1 holds.
(b) det $A(z)$ and det $\bar{A}(z)$ have $4N$ simple zeros outside and inside the unit circle C, respectively. None of them lies on the unit circle C.
(c) det $A(z)$ and det $\bar{A}(z)$ approach 1 rapidly as $|z| \to \infty$ and $z \to 0$, respectively.

Then, we can derive the following expansion for the sd-CNLS equations (2.30):

$$\begin{aligned}\log\det A(z) &= \sum_{n=1}^{\infty}\frac{1}{z^n}\left[\frac{1}{n}\sum_{j=1}^{4N}\left\{\left(\frac{1}{z_j^*}\right)^n - z_j^n\right\} + \frac{1}{2\pi\mathrm{i}}\oint_C w^{n-1}\log\det(A(w)\,\bar{A}(w))\,\mathrm{d}w\right]\\ &= \sum_{k=1}^{\infty}\frac{1}{z^{2k}}\left[\frac{1}{k}\sum_{j=1}^{2N}\left\{\left(\frac{1}{z_j^*}\right)^{2k} - z_j^{2k}\right\} + \frac{1}{\pi\mathrm{i}}\int_{C_R} w^{2k-1}\log\det(A(w)\,\bar{A}(w))\,\mathrm{d}w\right]\\ &= \sum_{k=1}^{\infty}\frac{1}{z^{2k}}\left[\frac{1}{k}\sum_{j=1}^{N}\left\{\left(\frac{1}{z_j}\right)^{2k} + \left(\frac{1}{z_j^*}\right)^{2k} - z_j^{2k} - z_j^{*\,2k}\right\}\right.\\ &\quad\left.+\frac{1}{\pi\mathrm{i}}\int_{C_{UR}}\left(w^{2k-1}+w^{-2k-1}\right)\log|\det A(w)|^2\,\mathrm{d}w\right].\end{aligned} \quad \text{(A.4)}$$

The coefficients of $1/z^{2k}$ $(k = 1, 2, \ldots)$ give an infinite number of the integrals of motion, which we call the *trace formulae*. Here, C_R and C_{UR} denote the right-half portion and the upper-right portion of the unit circle C, respectively, as is mentioned in section 3. It is recalled that $\boldsymbol{z}$ in (A.2) and z in (A.4) have the difference $\pi/4$ in their phases (see equation (3.4)). The determinant of $\bar{A}(z)$ is related to the determinant of $A(z)$ by (3.36). It can also be shown that $\det \mathcal{A}(z)$ and $\det \bar{A}(z)$ are connected by

$$\log\det\mathcal{A}(z) = -\sum_{n=-\infty}^{\infty}\log\det(I - Q_n R_n) + \log\det\bar{A}(z).$$

Thus, we can directly obtain expansions of $\log\det\bar{A}(z)$ and $\log\det\mathcal{A}(z)$ with respect to z from (A.4).

A derivation of (A.4) is omitted because it is analogous to that in the continuous theory [5, 56, 57]. Related results were also obtained by Kodama [58].

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